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THE ATTACK AND DEFENSE OF TARGETS BY MISSILES

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HOLLOMAN AIR FORCE BASE

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THE ATTACK AND DEFENSE OF TARGETS BY MISSILES

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FOREWORD

The author of this report, Dr William McEwen, is Professor of Mathematics and Chairman of the Division of Mathematics and Science at the University of Minnesota, Duluth, Minnesota; and a Consultant to the Directorate of Research Analyses, Air Force Office of Scientific Research.

The study was conducted during the period June through August 1961.

The work supports Project 5101, System Synthesis and Analysis, and represents the mathematical treatment of problems of general military interest.

ABSTRACT

This paper derives optimum methods of attacking and defending targets with missiles under certain assumptions as to the number of missiles, antimissiles, and targets. Both probability and game theory are used in the derivations, and the results obtained by both methods are shown to be substantially the same.

This report is approved for publication.

James H. Ritter

JAMES H. RITTER
Colonel, USAF
Director, Research Analyses Directorate

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THE ATTACK AND DEFENSE OF TARGETS BY MISSILES

I. INTRODUCTION

This paper will consider the problem in which a total of t targets are attacked by N missiles and defended by M antimissiles. Let the i^{th} target be attacked by n_i missiles, $n_i = 0, 1, 2, \ldots n$, and defended by m_i antimissiles, $m_i = 0, 1, 2, \ldots m$, and where

$$\sum_{i=1}^{t} m_i = M \qquad \text{and} \qquad \sum_{i=1}^{t} n_i = N .$$

The probability that an antimissile destroys a missile is taken as unity. That is, the i^{th} target survives if $m_i \geq n_i$, and is destroyed if $m_i < n_i$.

II. ASSUMPTIONS

N = total number of offensive missiles

M = total number of defensive antimissiles

t = total number of targets

$$m = \frac{2M}{t} \qquad\qquad n = \frac{2N}{t}$$

m and n are integers.

III. CONCLUSIONS

The offense should attack any single target with either

$$k,\ k+1,\ k+2,\ \ldots,\ \text{or } (n-k) \text{ missiles,}$$

and the defense should defend the same target with either

$$\ell,\ \ell+1,\ \ell+2,\ \ldots,\ \text{or } (m-\ell) \text{ antimissiles.}$$

The offense should choose to attack with exactly $k + i$ missiles with a frequency of q_i, where

$$\sum_{0}^{n-2k} q_i = 1 \qquad \text{and} \qquad q_i = q_{n-2k-i} \; .$$

Similarly, the defense should choose to defend with exactly $\ell + i$ antimissiles with a frequency of p_i, where

$$\sum_{0}^{m-2\ell} p_i = 1 \qquad \text{and} \qquad p_i = p_{m-2\ell-i} \; .$$

The values of k and ℓ are determined by the relationship between the total number of missiles available to the offense and defense.

If $m = n$ then $k = \ell = 0$

If $m > n$ then $k = 0$

$$\ell = \begin{cases} m - n \\ \text{or} \\ m - n - 1 \end{cases}$$

If $m < n$ then $k = \begin{cases} n - m \\ \text{or} \\ n - m - 1 \end{cases}$

$$\ell = 0$$

If $m \geq n$ the probability that a target is destroyed is

$$\frac{1}{2} \; \frac{n}{m + 1} \; ,$$

while if $m < n$ the probability is

$$\frac{1}{2} \; \frac{2n - m}{n + 1} \; .$$

These results show that an attempt to defend targets by destroying the attacking missiles will be ineffective unless the number of antimissiles available to the defense greatly exceeds the number of missiles available to the offense. They strongly suggest additional defensive measures.

IV. STATISTICAL THEORY

It is possible to attack this problem with statistical or game theory methods. Slightly different assumptions are necessary in each case, but the results, using either method, are in substantial agreement. The statistical results will be developed first, and the following assumptions are made:

a) The n_i missiles with which the offense attacks the i^{th} target do not exceed $\frac{2N}{t}$. That is

$$0 \leq n_i \leq n \qquad \text{where} \qquad n = \frac{2N}{t}$$

b) If q_k is the probability that $n_i = k$, then $q_k = \frac{1}{n+1}$, $k = 0, 1, 2, \ldots n$.

c) $0 \leq m_i \leq m$ where $m = \frac{2M}{t}$

d) If p_k is the probability that $m_i = k$, then $p_k = \frac{1}{m+1}$, $k = 0, 1, 2, \ldots m$.

e) m and n are integers. $M \geq t$, $N > t$

f) N is not the total number of missiles available to the offense, but the number that can be fired in a single salvo at the t targets. M is the number of antimissiles available to defend against this single salvo. The absolute total of missiles and antimissiles is assumed great enough for several salvos.

Some discussion of the reasons for the preceding assumption is perhaps in order. The reasoning will be stated in terms of the offense, but applies equally well to the defense.

Suppose the defense has exact knowledge as to the number of missiles which will attack a particular target. It can then prepare a perfect defense or, if the total number of antimissiles is insufficient, abandon the target in order to defend others more efficiently. Therefore, the offense must adopt some system of choosing the number of missiles with which to attack a particular target which will make it impossible for the defense to predict this number with certainty.

Also, since only N missiles are available in all, the system must guarantee that

$$\sum_{i=1}^{t} n_i \leq N ,$$

and for maximum effect the equal sign must hold. Assumptions a) and b) define such a system for the offense.

There are, of course, many other systems which would accomplish the same objective, and we will consider some of them in the course of this paper.

We now fix our attention on a single target and determine the probability that it will be destroyed. Since the offense can consist of 0, 1, 2 ... n missiles, the defense of 0, 1, 2, ... m antimissiles, there are $(m + 1)$ $(n + 1)$ equally likely cases to be considered. This is most conveniently done by means of the $(n + 1) \times (m + 1)$ matrix which follows. A zero in a cell represents a case in which the target survives; a one represents a case in which the target is destroyed.

Number of Antimissiles Used by Defense

Number of Missiles Used by Offense

	0	1	2	3	. . .	m
0	0	0	0	0	. . .	0
1	1	0	0	0	. . .	0
2	1	1	0	0	. . .	0
3	1	1	1	0	. . .	0
.	.	.	.	.	. . .	.
.	.	.	.	.	. . .	.
.	.	.	.	.	. . .	.
n	1	1	1	1	. . .	$\begin{cases} 1 & m < n \\ 0 & m \geq n \end{cases}$

In the case where $m < n$ there are i ones in the i^{th} row for $i = 0, 1, 2, \dots m$, and $(m + 1)$ ones in the remaining $n - m$ rows.

If q is the probability that the target is destroyed, and p the probability it survives, then

$$q = \frac{1 + 2 + 3 + \ldots + m + (n - m)(m + 1)}{(n + 1)(m + 1)}$$

$$= \frac{\frac{1}{2} m (m + 1) + (n - m)(m + 1)}{(n + 1)(m + 1)} \quad .$$

$$\left.\begin{array}{ll} & 1) \quad q = \frac{1}{2} \, \frac{2n - m}{n + 1} \\[2ex] \text{Since} & \quad p = 1 - q \\[2ex] & 2) \quad p = \frac{1}{2} \, \frac{m + 2}{n + 1} \end{array}\right\} \quad m < n$$

When $m \geq n$ there are i ones in the i^{th} row $i = 0, 1, 2, \ldots n$ and

$$q = \frac{1 + 2 + 3 + \ldots + n}{(n + 1)(m + 1)}$$

$$\left.\begin{array}{l} 3) \quad q = \frac{1}{2} \, \frac{n}{m + 1} \\[2ex] 4) \quad p = \frac{1}{2} \, \frac{2m - n + 2}{m + 1} \end{array}\right\} \quad m \geq n$$

The cases in which $m = n$ are particularly interesting. Here we have

$$\left.\begin{aligned} p &= \frac{1}{2} \; \frac{n+2}{n+1} \\ q &= \frac{1}{2} \; \frac{n}{n+1} \end{aligned}\right\} \quad m = n$$

Since it has been assumed that $\frac{n}{2} \cdot t = N$, n is always 2 or more and the minimum value of q is $\frac{1}{3}$ occurring when $n = 2$. For $n = 2, 3, 4, 5$ we have $q = \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{5}{12}$. As n increases q tends to $\frac{1}{2}$.

Even with $q = \frac{1}{3}$, the minimum value, if the offense is able to mount a series of attacks of the type described above, the probability of survival of a target decreases rapidly. If $p(k)$ is the probability that a target survives a series of k attacks, then,

$$p\,(k) = \left(\frac{2}{3}\right)^k .$$

For $k = 2$ the chance of survival is less than $1/2$, $k = 4$ less than $1/5$ and $k = 6$ less than $1/10$.

If $m \neq n$ we can write formulas 2) and 4) in the form

$$5) \quad m = 2\,[p\,n - (1 - p)] \qquad m < n$$

$$6) \quad m = \frac{1}{2\,(1 - p)}\,n - 1 \qquad m \geq n$$

Figure 1 is a graph of these equations for selected values of p. Since $N = t\left(\frac{n}{2}\right)$ and $M = t\left(\frac{m}{2}\right)$, the ratio $\frac{m}{n} = \frac{M}{N}$ gives the relationship between antimissiles and missiles to achieve the desired probability of survival or destruction of a target.

A more convenient form for equations 1) and 3) is obtained by setting $m = n - k \geq 0$. This yields

7) $$q = \frac{1}{2}\,\frac{n + k}{n + 1} \qquad 0 < k \leq n$$

8) $$q = \frac{1}{2}\,\frac{n}{n - k + 1} \qquad k \leq 0$$

These are plotted in Figure 2 for various values of the parameter k.

Let us now modify our original assumptions to read as follows:

a) $1 \leq n_i \leq n - 1 \qquad n = \frac{2N}{t}$

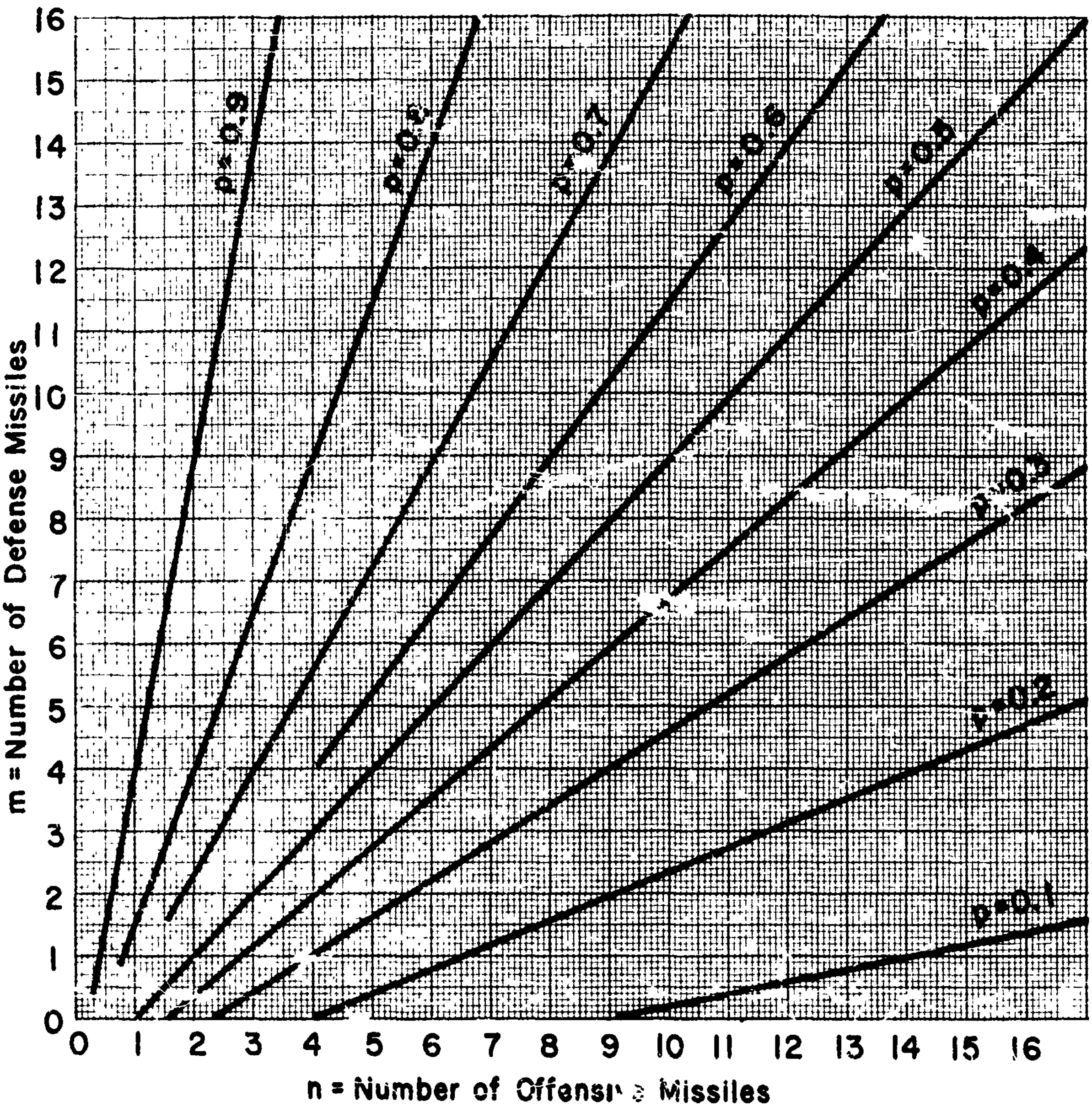
b) If q_k is the probability that $n_i = k$, $q_k = \frac{1}{n - 1}$,

$k = 1, 2, \ldots n - 1$

c) $1 \leq m_i \leq m - 1 \qquad m = \frac{2N}{t}$

d) $p_k = \frac{1}{m - 1} \qquad k = 1, 2, \ldots (m - 1)$

This is the case in which every target is attacked by at least one missile, and defended by at least one antimissile.



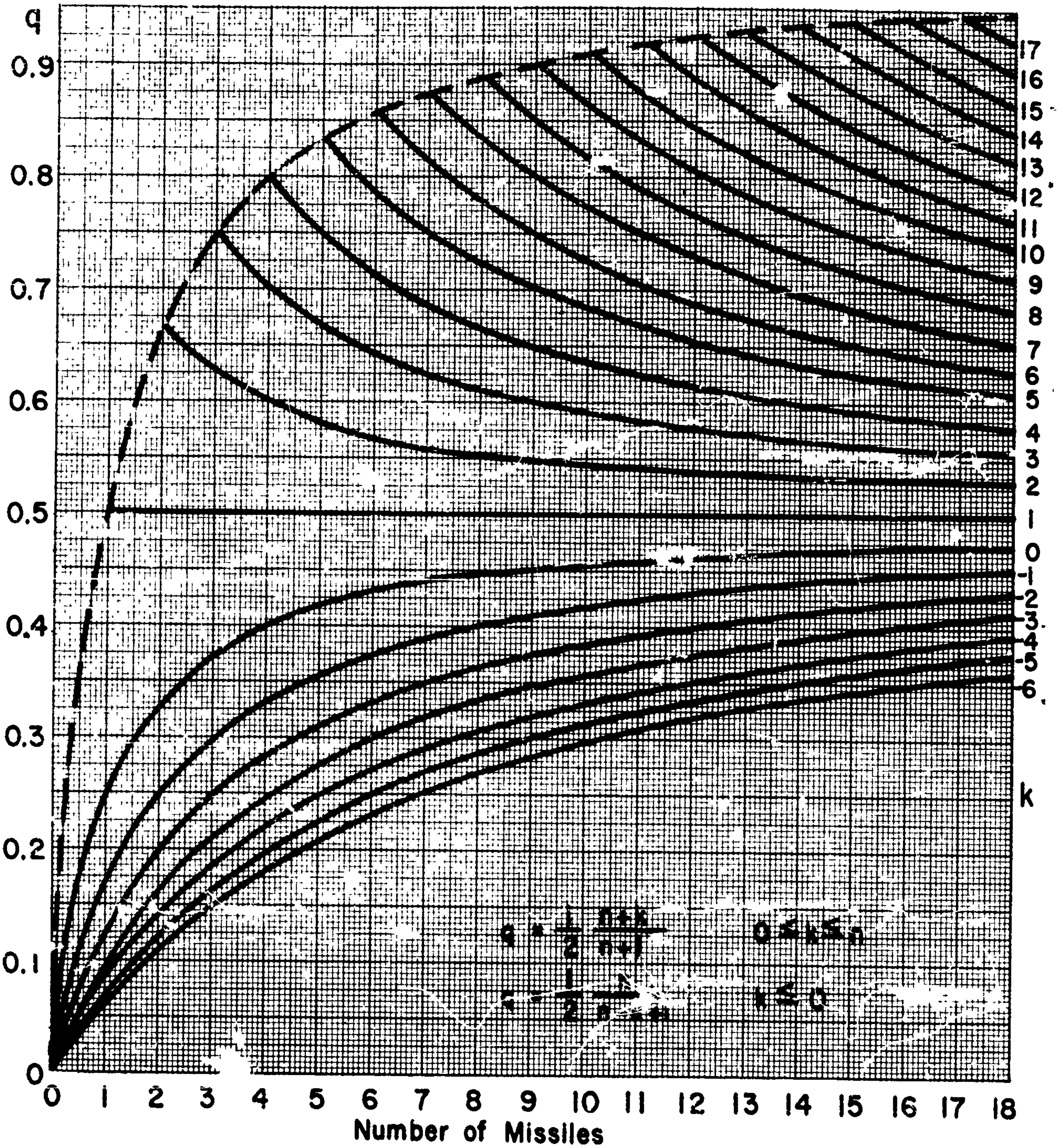
m = number of antimissiles available

0, 1, ... m equally likely to be used

n = number of missiles

0, 1, ... n equally likely to be used

Figure 1. Number of Antimissiles to Insure a Probability, p, of Survival of a Single Target.



n=number of attacking missiles available 0,1,2,...n are equally likely

m=n-k=number of defending missiles available

0,1,2,...(n-k) are equally likely

Figure 2. Probability a Single Target is Destroyed

Under these conditions equations 1) through 8) become

$$\left.\begin{array}{ll} 1') & q' = \frac{1}{2}\,\frac{2n - m - 2}{n - 1} \\[2ex] 2') & p' = \frac{1}{2}\,\frac{m}{n - 1} \end{array}\right\} \quad m < n$$

$$\left.\begin{array}{ll} 3') & q' = \frac{1}{2}\,\frac{n - 2}{m - 1} \\[2ex] 4') & p' = \frac{1}{2}\,\frac{2m - n}{m - 1} \end{array}\right\} \quad m \geq n$$

$$5') \quad m = 2pn - 2p \qquad m < n$$

$$6') \quad m = \frac{1}{2(1 - p)}\,n - \frac{p}{1 - p} \qquad m \geq n$$

$$7') \quad q' = \frac{1}{2}\,\frac{n + k - 2}{n - 1} \qquad 0 < k \leq n - 2$$

$$8') \quad q' = \frac{1}{2}\,\frac{n - 2}{n - k - 1} \qquad k \leq 0$$

These results have not been plotted separately, since they may be obtained from Figures 1 and 2 by increasing the indicated values of m and n by 2.

If $m < n + 1$, then $q < q'$ and $p > p'$, and if $m > n + 1$, then $q > q'$ and $p < p'$. That is, it is to the advantage of the side with the most weapons, if both sides use the

$$1, 2, \ldots (m - 1) \text{ or } (n - 1)$$

strategy. It can also be shown that if $m < n$, and the offense uses this strategy, the defense suffers the same losses using either the

$$1, 2, \ldots m-1 \quad \text{or the}$$

$$0, 1, 2, \ldots m$$

strategy. However, this and other results, will be shown by the consideration of the next problem.

We will now consider a more general problem of which the first two were special cases.

Assumptions

a) $k \leq n_i \leq n - k \qquad n = \frac{2N}{t}$

b) $q_i = \frac{1}{n - 2k + 1} \qquad i = k, k+1, \ldots (n-k)$

c) $\ell \leq m_i \leq n - \ell$

d) $p_i = \frac{1}{m - 2\ell + 1} \qquad i = \ell, \ell + 1, \ldots (m - \ell)$.

i. e., it is equally likely that a target be attacked by

$$k \text{ or } k+1 \text{ or } k+2 \ldots \text{ or } n-k$$

missiles, and it is also equally likely to be defended by

$$\ell \text{ or } \ell + 1 \text{ or } \ell + 2 \ldots . \text{ or } m - \ell$$

antimissiles.

Let p be the probability that a target survive and q be the probability that it is destroyed, $p + q = 1$. The offense would like to choose k so as to have q a maximum, while the defense would like to choose ℓ so that q is a minimum (p a maximum).

We will consider two cases; one in which the offense dominates, $n > m$, and one in which the defense dominates, $m > n$.

Case 1. $n > m$, $k > \ell$, $n - k > m - \ell$

Let $k - \ell = r > 0$ and again consider the $(n - 2k + 1) \times (m - 2\ell + 1)$ equally likely cases by means of a rectangular matrix

Number of Antimissiles

Number of Missiles

$$\begin{array}{c|ccccc}
 & \ell & \ell+1 & \ell+2 & \cdots & m-\ell \\
\hline
k & a_{k,\ell} & a_{k,\ell+1} & a_{k,\ell+2} & \cdots & a_{k,m-\ell} \\
k+1 & a_{k+1,\ell} & a_{k+1,\ell+1} & a_{k+1,\ell+2} & \cdots & a_{k+1,m-\ell} \\
k+2 & a_{k+2,\ell} & a_{k+2,\ell+1} & a_{k+2,\ell+2} & \cdots & a_{k+2,m-\ell} \\
\vdots & & & & & \\
n-k & a_{n-k,\ell} & a_{n-k,\ell+1} & a_{n-k,\ell+2} & \cdots & a_{n-k,m-\ell}
\end{array}$$

where
$$a_{ij} = 0 \qquad i \le j$$
$$= 1 \qquad i > j .$$

A zero represents a case in which the target survives, a one a case in which it is destroyed.

Since $k - \ell = r$

The first row contains r ones and $(m - 2\ell + 1 - r)$ - zeros

The second row contains $(r + 1)$ ones and $(m - 2\ell + 1 - r - 1)$ - zeros

The third row contains $(r + 2)$ ones and $(m - 2\ell + 1 - r - 2)$ - zeros

. . .

The number of zeros decreases by unity from row to row and there will be a row with one zero because $a_{n-k,\, m-\ell} = 1$ since $n - k > m - \ell$.

Therefore,

$$p = \frac{1 + 2 + 3 + \ldots + (m - 2\ell - r + 1)}{(n - 2k + 1)\ (m - 2\ell + 1)}$$

$$= \frac{1}{2}\ \frac{(m - 2\ell - r + 1)\ (m - 2\ell - r + 2)}{(n - 2k + 1)\ (m - 2\ell + 1)}$$

$$p = \frac{1}{2}\ \frac{(m - \ell - k + 1)\ (m - \ell - k + 2)}{(n - 2k + 1)\ (m - 2\ell + 1)}\ .$$

Taking $\frac{\partial}{\partial \ell}$ $(\log p)$ we obtain

$$\frac{1}{p}\ \frac{\partial p}{\partial \ell} = -\frac{1}{m - \ell - k + 1} - \frac{1}{m - \ell - k + 2} + \frac{2}{m - 2 - \ell + 1}\ .$$

Now $\dfrac{1}{m - \ell - k + 2} \geq \dfrac{1}{m - 2\ell + 1}$ if $k \geq \ell + 1$

or $k > \ell$

and $\dfrac{1}{m - \ell - k + 1} > \dfrac{1}{m - \ell - k + 2} \geq \dfrac{1}{m - 2\ell + 1}\ .$

Therefore $\frac{\partial p}{\partial \ell}$ is negative and the maximum value of p for $\ell \geq 0$ occurs when $\ell = 0$. Similarly

$$\frac{1}{p}\frac{\partial p}{\partial k} = -\frac{1}{m - \ell - k + 1} - \frac{1}{m - \ell - k + 2} + \frac{2}{n - 2k + 1} .$$

Now $\frac{1}{m - \ell - k + 2} \geq \frac{1}{n - 2k + 1}$

if

$$n - 2k + 1 \geq m - \ell - k + 2$$

$$n - k \geq m - \ell + 1$$

$$n - k > m - \ell$$

and

$$\frac{1}{m - \ell - k + 1} > \frac{1}{m - \ell - k + 2} \geq \frac{1}{n - 2k + 1} .$$

Therefore $\frac{\partial p}{\partial k}$ is also negative and since $\ell < k < n - m + \ell$ the minimum value of p occurs when $k = n - m - 1$.

We have proved, then, that if the offense dominates the defense the best strategy for the defense is to use

$$0, 1, 2, \ldots m$$

antimissiles with equal probability, while the offense should use the

$$k, k + 1, \ldots n - k$$

equal probability strategy with

$$k = n - m - 1 .$$

Case 2. Defense dominates the offense,

$$m > n, \quad \ell > k, \quad m - \ell > n - k .$$

If we again examine our probability matrix, we find

1st row contains all zeros

2nd row contains all zeros

r^{th} row contains all zeros

$(r + 1)^{st}$ row contains 1 one

$(r + 2)^{nd}$ row contains 2 ones

$[r + (n - 2k + 1 - r)]$ row contains $(n - 2k - r + 1)$ ones.

The last row actually contains exactly $n - k - r$ ones since

$$a_{n-k,\, m-\ell} = 0 \quad \text{because} \quad n - k < m - \ell .$$

$$\text{Therefore} \quad q = \frac{1 + 2 \; . \; . \; . + (n - 2k - r + 1)}{(n - 2k + 1)\,(m - 2\ell + 1)}$$

$$= \frac{1}{2}\,\frac{(n - 2k - r + 1)\,(n - k - r + 2)}{(n - 2k + 1)\,(m - 2\ell + 1)}$$

$$q = \frac{1}{2}\,\frac{(n - k - \ell + 1)\,(n - k - \ell + 2)}{(n - 2k + 1)\,(m - 2\ell + 1)} .$$

Proceeding as in Case 1, it is easy to show that $\frac{\partial q}{\partial \ell}$ and $\frac{\partial q}{\partial k}$ are both negative and therefore the maximum value of q for $k \geq 0$ occurs when $k = 0$, and the minimum value of q for $k < \ell < m - n + k$ occurs when $\ell = m - n - 1$.

This completes the proof that the side with the greater number of missiles should choose the

$$k,\ k+1,\ \ldots\ \begin{matrix} n - k \\ m - k \end{matrix}$$

strategy with $k = | m - n | - 1$, while the opponents should choose the

$$0,\ 1,\ 2\ \ldots\ \begin{matrix} m \\ n \end{matrix}$$

strategy.

If $\ell = 0$ and $k = n - m - 1$ are substituted in the expression

$$p = \frac{1}{2} \frac{(m - \ell - k + 1)\ (m - \ell - k + 2)}{(n - 2k + 1)\ (m - 2\ell + 1)} \quad n > m$$

it reduces to

$$p = \frac{1}{2} \frac{2m - n + 2}{m + 1} \ .$$

If $\ell = 0$ and $k = n - m$ are substitued in the same expressions it reduces to the same value. The reason is that if k is considered as a continuous variable the actual minimum occurs between

$$k = n - m - 1 \quad \text{and} \quad k = n - m$$

and the values of p for the integers on either side of the actual minimum turn out to be identical.

A similar statement applies to q , $n < m$. That is $\ell = m - n - 1$ and $k = 0$, and $\ell = m - n$ and $k = 0$ yields the same value of q , namely,

$$q = \frac{1}{2} \frac{2\ell - m + 2}{n + 1}$$

Therefore the side which dominates in the number of weapons may choose

$$k = |n - m| \text{ or } |n - m| - 1$$

and achieve the same results.

Now let us change the assumption that p_i and q_i are constant over the range of permissible values for i , and instead assume that

$$p_i = p_{m-i} \quad \text{and} \quad q_i = q_{n-i} .$$

We will go back to our original assumption that

$$0 \leq m_i \leq m \qquad m = \frac{2M}{t}$$

$$0 \leq n_i \leq n \qquad n = \frac{2N}{t} .$$

The previous problems are special cases of this one with particular values, for the p's and q's.

The probability that a target is defended by i antimissiles and attacked by j missiles is $p_i \; q_j$. The probability it is destroyed in this case is

$$a_{ij} \; p_i \; q_j$$

where $\qquad a_{ij} = 0 \quad \text{if} \quad i \geq j$

$\qquad\qquad = 1 \quad \text{if} \quad i < j$

Then the probability, q, that a target is destroyed may be expressed

$$q = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \; p_i \; q_j$$

$$= \sum_{i=0}^{m} \sum_{j=i+1}^{n} p_i \; q_j \; .$$

Let us first consider the case where $m < n$. Then

$$\text{a)} \quad q = p_0 \sum_{1}^{n} q_i + p_1 \sum_{2}^{n} q_i + \ldots + p_k \sum_{k+1}^{n} q_i + \ldots + p_m \sum_{m+1}^{n} q_i$$

If we make use of the fact $\sum_{1}^{n} q_i = 1$ we may re-write the expression a)

$$\text{b)} \quad q = p_0 \left(1 - \sum_{0}^{0} q_i \right) + p_1 \left(1 - \sum_{0}^{1} q_i \right) + p_2 \left(1 - \sum_{0}^{2} q_i \right)$$

$$+ \ldots + p_m \left(1 - \sum_{0}^{m} q_i \right) \; .$$

Now we reverse the order of the terms in a) and use the fact that $p_k = p_{m-k}$.

$$\text{c)} \quad q = p_0 \sum_{m+1}^{n} q_i + p_1 \sum_{m}^{n} q_i + p_2 \sum_{m-1}^{n} q_i + \ldots + p_m \sum_{1}^{n} q_i \; .$$

Because $q_k = q_{n-k}$ c) can be written

$$\text{d)}\quad q = p_o \sum_{0}^{n-m-1} q_i + p_1 \sum_{0}^{n-m} q_i + p_2 \sum_{0}^{n-m+1} q_i + \cdot \cdot + p_m \sum_{0}^{n-1} q_i$$

adding b) and d) we obtain

$$2q = p_o \left(1 + \sum_{1}^{n-m-1} q_i\right) + p_1 \left(1 + \sum_{2}^{n-m} q_i\right) + p_2 \left(1 + \sum_{3}^{n-m+1} q_i\right) + \cdots + p_m \left(1 + \sum_{m+1}^{n-1} q_i\right)$$

$$= 1 + p_o \sum_{1}^{n-m-1} q_i + p_1 \sum_{1}^{n-m-1} q_{i+1} + p_2 \sum_{1}^{n-m-1} q_{i+2} + \cdots + p_m \sum_{1}^{n-m-1} q_{i+m}$$

$$q = \frac{1}{2} \left[1 + \sum_{0}^{m} p_j \sum_{1}^{n-m-1} q_{j+i}\right]$$

$$q = \frac{1}{2} \left[1 + \sum_{j=o}^{m} \sum_{i=1}^{n-m-1} p_j\, q_{j+i}\right] \qquad m < n$$

It is interesting to note that when $q_k = \frac{1}{n+1}$, i.e., the offense uses the equal probability strategy of 0, 1, 2, ... n missiles then q reduces to

$$q = \frac{1}{2} \left[1 + \frac{n - m - 1}{n + 1}\right]$$

$$= \frac{1}{2} \frac{2n - m}{n + 1}$$

This is identical with expression 1) and again demonstrates that if the offense has more missiles than the defense, and chooses to use the equal probability strategy of 0, 1, 2, ... n missiles, the defense can do no better than adopt the same strategy. It is true of course that other strategies for the defense will do as well. In fact, in this case any strategy by the defense in which $p_k = p_{m-k}$ will yield the same value of q.

If $m \geq n$ the probability, q, that a target is destroyed is

$$q = \frac{1}{2}\left[1 - \sum_{j=0}^{n} \sum_{i=0}^{m-n} q_j \, p_{j+i}\right] \quad m \geq n \quad .$$

Here if $p_k = \frac{1}{m+1}$, q becomes

$$q = \frac{1}{2}\left[1 - \frac{m - n + 1}{m + 1}\right] = \frac{1}{2} \quad \frac{n}{m + 1} \qquad m \geq n$$

which is identical with expression 3).

The expressions for q when $m < n$ and $m \geq n$ show that if the side with the greater number of missiles chooses to adopt an equal probability strategy, then the opponent can do no better than adopt the same strategy.

However, if the side with the lesser number of missiles adopts the equal probability strategy, the opponent can obtain a better value for q than those given in 1) or 3).

Let us prove this for the case where the defense has more weapons than the offense $(m > n)$ and the offense chooses to attack with the

$$0, 1, 2, \dots n$$

equal probability strategy

$$\text{i.e.} \quad q_k = \frac{1}{n+1} \qquad k = 0, 1, 2, \dots n$$

Then

$$q = \frac{1}{2}\left[1 - \sum_{j=o}^{n} \sum_{i=o}^{m-n} q_j \, p_{j+i}\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n+1} \sum_{i=o}^{m-n} \sum_{j=o}^{n} p_{j+i}\right]$$

Now, the defense can choose

$$p_o = p_1 = \dots p_{m-n-1} = 0$$

$$p_{m-n} = p_{m-n+1} \dots = p_n = \frac{1}{2n - m + 1}$$

$$_{+1} = p_{m+2} = \dots = p_m = 0$$

Then $\displaystyle\sum_{j=o}^{n} p_{j+i} = 1 \qquad i = 0, 1, 2, \dots m - n$

and $$q = \frac{1}{2}\left[1 - \frac{m - n + 1}{n + 1}\right]$$

$$= \frac{1}{2}\,\frac{2n - m}{n + 1} \qquad m > n$$

while if

$$p_k = \frac{1}{m + 1} \qquad k = 0, 1, 2 \, \ldots \, m \, ,$$

we have shown that

$$q = \frac{1}{2}\,\frac{n}{m + 1} \qquad m \geq n \, .$$

Now

$$\frac{2n - m}{n + 1} < \frac{n}{m + 1}$$

whenever $m > n$, which proves our statement. The choice of

$$p_k = \frac{1}{2n - m + 1} \qquad k = m - n, m - n + 1, \ldots, n$$

is sufficient but not necessary since any choice which will make

$$\sum_{j=0}^{n} p_{j+i} = 1 \qquad i = 0, 1, 2, \ldots m - n$$

will serve the same purpose.

The solutions to the preceding problems make it evident that the defense is in an unenviable position. Even under the most optimistic

assumptions, i.e., every antimissile will destroy a missile, the number of antimissiles exceeds the number of missiles, the probability that a particular target is destroyed is

$$q = \frac{1}{2} \frac{n}{m + 1}$$

so that unless m is much greater than n (which appears unlikely) there will be considerable destruction of targets on each salvo.

As an example, suppose that $m + 1 = 3n$, then $q = \frac{1}{6}$.

In this case 4 salvos would destroy over 50 percent of the targets

It seems obvious that the defense of the targets must consist of more than an attempt to destroy the attacking missiles.

V. GAME THEORY

To apply game theory to the problem, it is necessary to change our assumptions slightly.

Let us consider a simple case first, namely one in which there are a total of n targets, n missiles, and n antimissiles, and that the strategies available to both offense and defense are the following:

> The offense may attack all the targets with one missile, $\frac{1}{2}$ the targets with 2 missiles, $\frac{1}{3}$ with 3 missiles, and so on.
>
> The defense will defend with a similar strategy.

A target will survive if it is defended by at least as many antimissiles as the number of attacking missiles and is destroyed otherwise.

Now suppose the offense attacks $\frac{1}{k} n$ targets with k missiles each and the defense defends $\frac{1}{\ell} n$ targets with ℓ missiles each. If the targets to be attacked and those to be defended are chosen at random, then the k missiles will destroy $\frac{1}{k} n$ targets if $k > \ell$, but only $\frac{1}{k} (1 - \frac{1}{\ell}) n$

targets if $k \leq \ell$, where $(1 - \frac{1}{\ell})\, n$ is the number of undefended targets. Since $\frac{1}{k}$ and $\frac{1}{k}(1 - \frac{1}{\ell})$ represent the fraction of the targets destroyed we will call them the payoff to the offense in each case. The matrix which follows gives this payoff for the various combinations of strategies available to the offense and defense.

Let $S_k^{\,1}$ indicate the strategy of attacking (or defending) $\frac{n}{k}$ targets with k missiles, and let p_k be the probability the defense uses the strategy $S_k^{\,1}$ and q_k the probability for the offense.

Now the defense would like to choose the p's so that the payoff is a minimum while the offense will choose q's so as to make it a maximum. Let us assume that there is some number v which represents both the minimum payoff the defense can achieve, and the maximum payoff available to the offense.

Defense

$$
\text{Offense}\quad
\begin{array}{c|cccccc|c}
 & S_1^1 & S_2^1 & S_3^1 & \cdots & S_k^1 & \cdots & \\
\hline
S_1^1 & 0 & \frac{1}{2} & \frac{2}{3} & & \frac{k-1}{k} & & q_1 \\
S_2^1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{3} & & \frac{k-1}{2k} & & q_2 \\
S_3^1 & \frac{1}{3} & \frac{1}{3} & \frac{2}{9} & & \frac{k-1}{3k} & & q_3 \\
\vdots & & & & & & & \\
S_k^1 & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & & \frac{k-1}{k^2} & & q_k \\
\vdots & & & & & & & \\
\hline
 & p_1 & p_2 & p_3 & & p_k & &
\end{array}
$$

S_k^1 indicates the strategy of attacking (or defending) $\frac{n}{k}$ targets with k missiles.

p_k probability defense uses the strategy S_k^1

q_k probability offense uses the strategy S_k^1

If there is such a number, the following equations must be satisfied:

$$0\,p_1 + \frac{1}{2}p_2 + \frac{2}{3}p_3 + \;\ldots\; + \frac{k-1}{k}p_k + \;\ldots\; = v$$

$$\frac{1}{2}p_1 + \frac{1}{4}p_2 + \frac{1}{3}p_3 + \;\ldots\; + \frac{k-1}{2\,k}p_k + \;\ldots\; = v$$

$$\frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{2}{9}p_3 + \;\ldots\; + \frac{k-1}{3\,K}p_k + \;\ldots\; = v$$

$$\frac{1}{k}p_1 + \frac{1}{k}p_2 + \frac{1}{k}p_3 + \;\ldots\; + \frac{1}{k}p_{k-1} + \frac{k-1}{k^2}p_k + \;\ldots\; = v$$

$$p_1 + p_2 + p_3 + \;\ldots\; + p_k \;\ldots\; = 1$$

$$0\,q_1 + \frac{1}{2}q_2 + \frac{1}{3}q_3 + \;\ldots\quad\ldots\; + \frac{1}{k}q_k + \;\ldots\; = v$$

$$\frac{1}{2}q_1 + \frac{1}{4}q_2 + \frac{1}{3}q_3 + \;\ldots\quad\ldots\; + \frac{1}{k}q_k + \;\ldots\; = v$$

$$\frac{2}{3}q_1 + \frac{1}{3}q_2 + \frac{2}{9}q_3 + \;\ldots\quad\ldots\; + \frac{1}{k}q_k + \;\ldots\; = v$$

$$\frac{k-1}{k}q_1 + \frac{k-1}{2\,k}q_2 + \frac{k-1}{3\,k}q_3 + \;\ldots\; + \frac{k-1}{(k-1)k}q_{k-1} + \frac{k-1}{k^2}q_k + \;\ldots\; = v$$

$$q_1 + q_2 + q_3 + \;\ldots\; + q_k + \;\ldots\; = 1$$

In the set of equations for the p's multiply the first equation by one, the second by 2, the k^{th} by k, etc. Then subtract the $(k-1)^{st}$ equation from the k^{th}, the $(k-2)^{nd}$ from the $(k-1)^{st}$, etc. to obtain

$$0\,p_1 + \frac{1}{2}p_2 + \frac{2}{3}p_3 + \dots + \frac{k-1}{k}p_k + \dots = v$$

$$p_1 + 0 + 0 + \dots + 0 + \dots = v$$

$$0 + \frac{1}{2}p_2 + 0 + \dots + 0 + \dots = v$$

$$0 + 0 + \frac{1}{3}p_3 + 0 \dots + 0 + \dots = v$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$0 + 0 + 0 + 0 + \frac{1}{k-1}p_{k-1} + 0 + \dots = v$$

A formal solution is then:

$$p_1 = v, \; p_2 = 2v, \; \dots \; p_{k-1} = (k-1)v$$

with p_k being determined from the first equation. However, the restriction that all the p's are non-negative, combined with the first equation requires that $k = 2$, and

$$p_1 = v \quad p_2 = 2v .$$

Since $p_1 + p_2 = 1$ the value of the game $v = \frac{1}{3}$. Therefore, if the defense adopts the $S_1{}^1$ strategy with a probability of $\frac{1}{3}$ and the $S_2{}^1$ strategy with a probability of $\frac{2}{3}$, the payoff to the offense can be limited to $\frac{1}{3}$ of the targets. This is in exact agreement with the results under probability theory when $m = n = 2$, as it should be since the strategies in both cases are identical.

The q's can be determined in a similar manner, but a direct examination of the payoff matrix shows that if the offense adopts the $S_1{}^1$ and $S_2{}^1$ strategies with probabilities of $\frac{1}{3}$ and $\frac{2}{3}$, respectively, it can achieve a payoff of $\frac{1}{3}$, and no better, for none of the payoff values in the first two columns, other than the first two rows, exceed $\frac{1}{3}$.

Now consider the case where both the defense and offense have k n missiles and both use the strategies S_k^k S_{k+1}^k . . . $S_{k+\ell}^k$, where S_{k+i}^k indicates the strategy of attacking (or defending) $\frac{k\,n}{k+i}$ targets with $(k + i)$ missiles. Let p_i and q_i be the probabilities that the defense and offense, respectively, use the strategy S_{k+i}^k.

Again let us assume that there is some number v which represents the smallest payoff that can be managed by the defense, and the largest payoff available to the offense.

If v exists, the equations which follow the payoff matrix must be satisfied.

Defense

Offense	S^k_k	S^k_{k+1}	S^k_{k+2}	$\cdots$	$S^k_{k+\ell-2}$	$S^k_{k+\ell-1}$	$S^k_{k+\ell}$
S^k_k	0	$\frac{1}{k+1}$	$\frac{2}{k+2}$	$\cdots$	$\frac{\ell-2}{k+\ell-2}$	$\frac{\ell-1}{k+\ell-1}$	$\frac{\ell}{k+\ell}$
S^k_{k+1}	$\frac{k}{k+1}$	$\frac{k}{(k+1)^2}$	$\frac{2k}{(k+1)(k+2)}$	$\cdots$	$\frac{k(\ell-2)}{(k+1)(k+\ell-2)}$	$\frac{k(\ell-1)}{(k+1)(k+\ell-1)}$	$\frac{k\ell}{(k+1)(k+\ell)}$
S^k_{k+2}	$\frac{k}{k+2}$	$\frac{k}{k+2}$	$\frac{2k}{(k+2)^2}$	$\cdots$	$\frac{k(\ell-2)}{(k+2)(k+\ell-2)}$	$\frac{k(\ell-1)}{(k+2)(k+\ell-1)}$	$\frac{k\ell}{(k+2)(k+\ell)}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$S^k_{k+\ell-2}$	$\frac{k}{k+\ell-2}$	$\frac{k}{k+\ell-2}$	$\frac{k}{k+\ell-2}$	$\cdots$	$\frac{k(\ell-2)}{(k+\ell-2)^2}$	$\frac{k(\ell-1)}{(k+\ell-2)(k+\ell-1)}$	$\frac{k\ell}{(k+\ell-2)(k+\ell)}$
$S^k_{k+\ell-2}$	$\frac{k}{k+\ell-1}$	$\frac{k}{k+\ell-1}$	$\frac{k}{k+\ell-1}$	$\cdots$	$\frac{k}{k+\ell-1}$	$\frac{k(\ell-1)}{(k+\ell-1)^2}$	$\frac{k\ell}{(k+\ell-1)(k+\ell)}$
$S^k_{k+\ell}$	$\frac{k}{k+\ell}$	$\frac{k}{k+\ell}$	$\frac{k}{k+\ell}$	$\cdots$	$\frac{k}{k+\ell}$	$\frac{k}{k+\ell}$	$\frac{k\ell}{(k+\ell)^2}$

S^k_{k+i} indicates the strategy of attacking (or defending) $\frac{kn}{k+i}$ targets with $(k+i)$ missiles.

$$0 \cdot p_0 + \frac{1}{k+1} p_1 + \frac{2}{k+2} p_2 + \cdots$$

$$+ \frac{\ell - 2}{(k + \ell - 2)} p_{\ell-2} + \frac{\ell - 1}{k + \ell - 1} p_{\ell-1} + \frac{\ell}{k + \ell} p_\ell = v$$

$$\frac{k}{k+1} p_0 + \frac{k}{(k+1)^2} p_1 + \frac{2k}{(k+1)(k+2)} p_2 + \cdots$$

$$+ \frac{k(\ell - 2)}{(k+1)(k + \ell - 2)} p_{\ell-2} + \frac{k(\ell - 1)}{(k+1)(k + \ell - 1)} p_{\ell-1} + \frac{k\ell}{(k+1)(k + \ell)} p_\ell = v$$

$$\frac{k}{k+2} p_0 + \frac{k}{k+1} p_1 + \frac{2k}{(k+2)^2} p_2 + \cdots$$

$$+ \frac{k(\ell - 2)}{(k+2)(k + \ell - 2)} p_{\ell-2} + \frac{k(\ell - 1)}{(k+2)(k + \ell - 1)} p_{\ell-1} + \frac{k\ell}{(k+2)(k + \ell)} p_\ell = v$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{k}{k + \ell - 2} p_0 + \frac{k}{k + \ell - 2} p_1 + \frac{k}{k + \ell - 2} p_2 + \cdots$$

$$+ \frac{k(\ell - 2)}{(k + \ell - 2)^2} p_{\ell-2} + \frac{k(\ell - 1)}{(k + \ell - 2)(k + \ell - 1)} p_{\ell-1} + \frac{k\ell}{(k + \ell - 2)(k + \ell)} p_\ell = v$$

$$\frac{k}{k+\ell-1} p_0 + \frac{k}{k+\ell-1} p_1 + \frac{k}{k+\ell-1} p_2 + \cdots$$

$$+ \frac{k}{k+\ell-1} p_{\ell-2} + \frac{k(\ell-1)}{(k+\ell-1)^2} p_{\ell-1} + \frac{k\ell}{(k+\ell-1)(k+\ell)} p_\ell = v$$

$$\frac{k}{k+\ell} p_0 + \frac{k}{k+\ell} p_1 + \frac{k}{k+\ell} p_2 + \cdots$$

$$+ \frac{k}{k+\ell} p_{\ell-2} + \frac{k}{k+\ell} p_{\ell-1} + \frac{k}{(k+\ell)^2} p_\ell = v$$

$$p_0 + p_1 + \cdots + p_\ell = 1$$

$$0 \cdot q_0 + \frac{k}{k+1} q_1 + \frac{k}{k+2} q_2 + \cdots$$

$$+ \frac{k}{k+\ell-2} q_{\ell-2} + \frac{k}{k+\ell-1} q_{\ell-1} + \frac{k}{k+\ell} q_\ell = v$$

$$\frac{k}{k(k+1)} q_0 + \frac{k}{(k+1)^2} q_1 + \frac{k}{k+2} q_2 + \cdots$$

$$+ \frac{k}{k+\ell-2} q_{\ell-2} + \frac{k}{k+\ell-1} q_{\ell-1} + \frac{k}{k+\ell} q_\ell = v$$

$$\frac{2k}{k(k+2)} q_0 + \frac{2k}{(k+1)(k+2)} q_1 + \frac{2k}{(k+2)^2} q_2 + \cdots$$

$$+ \frac{k}{k+\ell-2} q_{\ell-2} + \frac{k}{k+\ell-1} q_{\ell-1} + \frac{k}{k+\ell} q_\ell = v$$

. . . .

$$\frac{(\ell-2)k}{k(k+\ell-2)} q_0 + \frac{(\ell-2)k}{(k+1)(k+\ell-2)} q_1 + \frac{(\ell-2)k}{(k+2)(k+\ell-2)} q_2 + \cdots$$

$$\frac{(\ell-2)k}{(k+\ell-2)^2} q_{\ell-2} + \frac{k}{k+\ell-1} q_{\ell-1} + \frac{\;}{k+\ell} q_\ell = v$$

$$\frac{(\ell-1)k}{k(k+\ell-1)} q_0 + \frac{(\ell-1)k}{(k+1)(k+\ell-1)} q_1 + \frac{(\ell-1)k}{(k+2)(k+\ell-1)} q_2 + \cdots$$

$$\frac{(\ell-1)k}{(k+\ell-2)(k+\ell-1)} q_{\ell-2} + \frac{(\ell-1)k}{(k+\ell-1)^2} q_{\ell-1} + \frac{k}{k+\ell} q_\ell = v$$

$$\frac{\ell k}{k(k+\ell)} q_0 + \frac{\ell k}{(k+1)(k+\ell)} q_1 + \frac{\ell k}{(k+2)(k+\ell)} q_2 + \cdots$$

$$+ \frac{\ell k}{(k+\ell-2)(k+\ell)} q_{\ell-2} + \frac{\ell k}{(k+\ell-1)(k+\ell)} q_{\ell-1} + \frac{\ell k}{(k+\ell)^2} q_\ell = v$$

$$q_0 + q_1 + q_2 + \cdots + q_\ell = 1$$

In the set of equations for the p's multiply the first by k, the second by k + 1, etc. Then subtract the ℓ^{th} from the $(\ell + 1)^{st}$, the $(\ell - 1)^{st}$ from the ℓ^{th}, etc. This will yield values for

$$p_0, p_1, p_2 \cdots p_{\ell-1}$$

while the value of p_ℓ can be obtained by substituting in the first equation

$$1) \quad \sum_{i=1}^{\ell} \frac{i}{k+i} p_i = v$$

The values obtained for the p's are

$$p_i = \frac{k+i}{k^2} v \qquad k = 0, 1, 2, 3 \cdots (\ell - 1)$$

Substituting in 1) we have

$$\sum_{i=1}^{\ell-1} \frac{i}{k^2} v + \frac{\ell}{k+\ell} p_\ell = v$$

$$\frac{1}{2} \frac{\ell(\ell-1)v}{k^2} + \frac{\ell}{k+\ell} p_\ell = v$$

$$p_\ell = \frac{k+\ell}{\ell}\left[1 - \frac{\ell(\ell-1)}{2k^2}\right] v$$

Since $p_\ell \geq 0$, $\ell(\ell-1) \leq 2k^2$, so that ℓ is the largest integer which satisfies the above inequality.

To determine v we make use of the fact that

$$\sum_{i=0}^{\ell} p_i = 1$$

$$\sum_{i=0}^{\ell-1} \frac{k+i}{k^2} v + \frac{k+\ell}{\ell}\left[1 - \frac{\ell(\ell-1)}{2k^2}\right] v = 1$$

$$v = \frac{2k\ell}{2k^2 + 2k\ell + \ell^2 + \ell}$$

When the equations for the q's are solved the following results are obtained

$$q_0 = k \frac{v}{\ell}$$

$$q_i = \frac{k+i}{k} \frac{v}{\ell} \quad i = 1, 2, \ldots \ell$$

where ℓ in this case is not restricted.

Since $\sum_{i=0}^{\ell} q_i = 1$ we have

$$\frac{v}{\ell}\left[k + \sum_{i=1}^{\ell} \frac{k+i}{k}\right] = 1$$

or $$v = \frac{2k\ell}{2k^2 + 2k\ell + \ell^2 + \ell}$$

Now the offense would like to choose ℓ so that v is a maximum.

Since
$$\frac{dv}{d\ell} = \frac{2k\,(2k^2 - \ell^2)}{2k^2 + 2k\ell + \ell^2 + \ell}$$

$$\ell = k\sqrt{2}$$

will yield the maximum, and it would appear that in some cases, at least, the offense would choose a value of ℓ at least one greater than that chosen by the defense. However, a direct examination of the payoff matrix shows that if the defense adopts the strategies S^k_{ki} S^k_{k+1} . . . $S^k_{k+\ell}$ with ℓ being the largest integer such that $\ell\,(\ell - 1) \leq 2k^2$ the defensive strategy $S^k_{k+\ell+1}$ has a payoff of exactly

$$\frac{k}{k + \ell + 1}$$

and
$$\frac{k}{k + \ell + 1} < \frac{2k\ell}{2k^2 + 2k\ell + \ell^2 + \ell}$$

whenever $\ell\,(\ell + 1) > 2k^2$, which is true since the defense has chosen ℓ as the largest integer for which

$$\ell\,(\ell - 1) \leq 2k^2 .$$

Therefore, the value of ℓ which yields a maximum v to the offense is the same as that which minimizes v for the defense.

The strategies S^k_{k+i} in which a target is attacked (or defended) by $\frac{k\,n}{k + i}$ missiles are probably not very realistic for values of k other than unity. As something which more nearly approximates an actual situation,

let us consider the case where an average of k missiles (or antimissiles) is available per target and let the strategies be as follows:

$\frac{t}{2\ell + 1}$ targets are attacked (or defended) by $k + \ell$ missiles

$\frac{t}{2\ell + 1}$ targets are attacked (or defended) by $k + \ell - 1$ missiles

. . . .

$\frac{t}{2\ell + 1}$ targets are attacked (or defended) by k missiles

. . . .

$\frac{t}{2\ell + 1}$ targets are attacked (or defended) by $k - \ell + 1$ missiles

$\frac{t}{2\ell + 1}$ targets are attacked (or defended) by $k - \ell$ missiles

The payoff matrix for the offense in this case becomes:

$$
\begin{array}{cc|cccccc}
 & & & & \text{Defense} & & & \\
 & & p_1 & p_2 & p_3 & p_4 & \cdots & p_\ell \\
\hline
 & p_1 & 0 & \frac{1}{3} & \frac{2}{5} & \frac{3}{7} & \cdots & \frac{\ell}{2\ell+1} \\
 & p_2 & \frac{1}{3} & \frac{1}{3} & \frac{2}{5} & \frac{3}{7} & \cdots & \frac{\ell}{2\ell+1} \\
\text{Offense} & p_3 & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{3}{7} & \cdots & \frac{\ell}{2\ell+1} \\
 & p_4 & \frac{3}{7} & \frac{3}{7} & \frac{3}{7} & \frac{3}{7} & \cdots & \frac{\ell}{2\ell+1} \\
 & \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & p_\ell & \frac{\ell}{2\ell+1} & \frac{\ell}{2\ell+1} & \frac{\ell}{2\ell+1} & \frac{\ell}{2\ell+1} & \cdots & \frac{\ell}{2\ell+1}
\end{array}
$$

The value of the game to the offense is $\frac{\ell}{2\ell+1}$, which will be a maximum when $\ell = k$. Then

$$v = \frac{k}{2k+1} .$$

Since p_k represents the strategy of attacking (or defending) with 0, 1, 2, . . . 2k missiles with equal probability, this is another instance which

shows that if the defense and offense have equal numbers of missiles, neither can do better than to adopt this strategy. Setting $k = \frac{n}{2}$, v reduces to $v = \frac{1}{2} \frac{n}{n+1}$, the expression derived for q under probability theory.

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THE ATTACK AND DEFENSE OF TARGETS BY MISSILES, by W. R. McEwen.
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